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SOME DIOPHANTINE APPROXIMATIONS CONNECTED WITH QUADRATIC SURDS

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1. Let θ be a positive irrational number, and let $a_1, a_2 \ldots$ be an increasing sequence of positive integers tending to infinity. The distribution of the set of numbers formed by the fractional parts of $a_n\theta$ is a very interesting problem. The classical results in this subject were established by Dirichlet and Kronecker [1]. In recent years the subject of Diophantine Approximations, of which the above is one of the problems, has attracted much attention not only for its own sake, but also for the applications it has found in the Analytic Theory of Numbers. Hardy, Littlewood and Bohr have contributed largely to these later developments [2].

2. Let us denote, as is usual in this subject, the fractional part of x by (x). A very elementary result which can be established, either by what is known as Dirichlet's principle, or by the theory of continued fractions, is that, if we take $\alpha_n = n$, the set of numbers $(n\theta)$ is dense everywhere in the unit segment (0, 1). If, however, we consider a sequence α_n which tends to infinity with great rapidity the set of numbers $(a_n\theta)$ may not be dense everywhere. For example, if $\theta = e$ and $a_n = n!$ then

$$(n!\,e) \rightarrow 0$$

(1) Kronecker: *Werke*, Vol III, p. 49. Dirichlet, *Werke*, Vol. I, p. 635. See also Minkowski, *Diophantische Approximationen.*

(2) G. H. Hardy and J.E. Littlewood: Some Problems of Diophantine Approximations, *Acta Mathematica*, Vol. XXXVII, 1914.

H. Bohr: Zur Theorie der Riemannschen Zetafunktion im Kritischen Streifen, *Acta Mathematica*, Vol. XL, 1916.

H. Bohr and R. Courant; Neue Anwendungen der Theorie der Diophantische Approximationen auf die Riemannsche Zetafunktion, *Crelles Journal*, 1914.

as $n \to \infty$, as is easily verified by using the infinite series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots;$$

so that the only limiting point of the set $(n!\,e)$ is zero [3].

3. The object of this note is to obtain some results regarding the distribution of the set of numbers $(q_n^2\,\theta)$, where θ is a quadratic surd and $\frac{p_n}{q_n}$ is the nth convergent in the continued fraction for θ. It is proved here that the limiting points of the set are finite in number and can be easily calculated.

4. We shall begin by considering the set of numbers

$$(1) \qquad q_n^2 \left| \theta - \frac{p_n}{q_n} \right|,$$

whose distribution is closely related to that of $(q_n^2\,\theta)$.

In considering the distribution of the set (1) there is no loss of generality, if we confine ourselves to quadratic surds whose continued fractions are purely recurring. To prove this, we observe first that if, θ and ϕ are two *equivalent* irrationals, that is to say, if they are connected by a relation of the form

$$\theta = \frac{A + B\phi}{C + D\phi},$$

where A, B, C, D are integers such that $AD - BC = \pm 1$, then the partial quotients in the continued fractions for θ and ϕ are ultimately identical.

Let $$\theta = b_1 + \frac{1}{b_2 +} \cdots \frac{1}{b_n +} \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots$$

and $$\phi = c_1 + \frac{1}{c_2 +} \cdots \frac{1}{c_m +} \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots,$$

and let $\frac{p_s}{q_s}$ and $\frac{P_s}{Q_s}$ denote the s^{th} convergents in these continued fractions. We shall prove that, as $r \to \infty$,

$$(2) \qquad q_{n+r}^2 \left| \theta - \frac{p_{n+r}}{q_{n+r}} \right| \sim Q_{m+r}^2 \left| \phi - \frac{P_{m+r}}{Q_{m+r}} \right|$$

from which it follows that the two sets of numbers

$$q_s^2 \left| \theta - \frac{p_s}{q_s} \right|,$$

$$Q_s^2 \left| \phi - \frac{P_s}{Q_s} \right|$$

(3) For other examples, see the paper of Hardy and Littlewood referred to above.

are distributed similarly ; and so in the case of quadratic surds there is no loss of generality in supposing the continued fraction to be purely recurring.

5. The proof of (2) is easy. We shall write for simplicity p, q, P, Q for p_{n+r}, q_{n+r}, P_{m+r}, Q_{m+r} ; so that, instead of $r \to \infty$, we may say q, $Q \to \infty$. We have

$$(3) \qquad \frac{p}{q} = \frac{A\frac{P}{Q} + B}{C\frac{P}{Q} + D} = \frac{AP + BQ}{CP + DQ},$$

where $AD - BC = \pm 1$. It is easily seen that the last fraction on the right hand side is in its lowest terms ; and so

$$p = AP + BQ, \; q = CP + DQ.$$

Now $$q^2 \left| \theta - \frac{p}{q} \right| = q^2 \left| \frac{A\phi + B}{C\phi + D} - \frac{AP+BQ}{CP+DQ} \right|$$

$$= \frac{| \phi Q - P | \, (CP + DQ)}{C\phi + D}$$

by making use of $| AD - BC | = 1$ and (3). Therefore

$$\frac{q^2 \left| \theta - \frac{p}{q} \right|}{Q^2 \left| \phi - \frac{P}{Q} \right|} = \frac{C\frac{P}{Q} + D}{C\phi + D} \to 1,$$

since $$\frac{P}{Q} \to \phi.$$

6. We shall now proceed to the proof of the main result of this paper. Let

$$\theta_1 = a_1 + \frac{1}{a_2 +} \cdots\cdots\cdots \frac{1}{a_n}$$
$$\qquad\qquad\quad * \qquad\qquad\qquad *$$

be a quadratic surd whose continued fraction is purely recurring. We shall adopt the notation

$$a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n} \equiv [a_1, a_2, \ldots\ldots\ldots a_n].$$
$$* \qquad\qquad\quad *$$

Let $\frac{p_s}{q_s}$ be the s^{th} convergent of θ_1. Then

$$\theta_1 = a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_n +} \frac{1}{\theta_1} = \frac{\theta_1 p_n + p_{n-1}}{\theta_1 q_n + q_{n-1}},$$

$$\left| \theta_1 - \frac{p_n}{q_n} \right| = \left| \frac{\theta_1 p_n + p_{n-1}}{\theta_1 q_n + q_{n-1}} - \frac{p_n}{q_n} \right|$$

$$= \frac{1}{q_n(\theta_1 q_n + q_{n-1})} = \frac{1}{q_n^2} \cdot \frac{1}{\theta_1 + \frac{q_{n-1}}{q_n}}.$$

Similarly $$\left| \theta_1 - \frac{p_{2n}}{q_{2n}} \right| = \frac{1}{q^2_{2n}} \cdot \frac{1}{\theta_1 + \frac{q_{2n-1}}{q_{2n}}}$$

and generally $$\left| \theta_1 - \frac{p_{sn}}{q_{sn}} \right| = \frac{1}{q^2_{sn}} \frac{1}{\theta_1 + \frac{q_{sn-1}}{q_{sn}}}.$$

Now $$\frac{q_{n-1}}{q_n} = \frac{1}{a_n +} \frac{1}{a_{n-1} +} \cdots \frac{1}{a_2},$$

$$\frac{q_{2n-1}}{q_{2n}} = \frac{1}{a_n +} \frac{1}{a_{n-1} +} \cdots\cdots \frac{1}{a_2 +} \frac{1}{a_1 +} \frac{1}{a_n +} \frac{1}{a_{n-1} +} \cdots\cdots \frac{1}{a_2},$$

and so on. It is easy to see that as $s \to \infty$ through integral values

$$\frac{q_{sn}}{q_{sn-1}} \to [a_n, a_{n-1}, \ldots a_1] \equiv -\frac{1}{\bar{\theta}_1}, \text{ say.}$$

Hence $$q^2_{sn} \left| \theta_1 - \frac{p_{sn}}{q_{sn}} \right| \to \frac{1}{\theta_1 - \bar{\theta}_1},$$

as $s \to \infty$.

If we start with the identity

$$\theta_1 = a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_n +} \frac{1}{a_1 +} \frac{1}{\theta_2},$$

where $$\theta_2 = [a_2, a_3, \ldots a_n, a_1],$$

and argue as above, we obtain the result that as $s \to \infty$

$$q^2_{sn+1} \left| \theta_1 - \frac{p_{sn+1}}{q_{sn+1}} \right| \to \frac{1}{\theta_2 - \bar{\theta}_2},$$

where $[a_1, a_n, a_{n-1} \ldots a_2] = -\dfrac{1}{\bar{\theta}_2}$.

Generally, if r is an integer such that $1 \leq r \leq n$, and if

$$\theta_r = [a_r, a_{r+1} \dots a_n, a_1, a_2 \dots\dots a_{r-1}],$$

then
$$q^2_{sn+r-1} \left| \theta_1 - \frac{p_{sn+r-1}}{q_{sn+r-1}} \right| \to \frac{1}{\theta_r - \bar{\theta}_r},$$

as $s \to \infty$; $\bar{\theta}_r$ being defined by

$$[a_{r-1}, a^{r-2} \dots a_1, a_n, a_{n-1} \dots a_r] = -\frac{1}{\bar{\theta}_r}.$$

Hence the only limiting points of the set of numbers

$$q_r^2 \left| \theta_1 - \frac{p_r}{q_r} \right|$$

are
$$\frac{1}{\theta_1 - \bar{\theta}_1}, \frac{1}{\theta_2 - \bar{\theta}_2} \dots\dots \frac{1}{\theta_n - \bar{\theta}_n}.$$

7. We can now obtain the limiting points of the set $(q^2_n\, \theta)$ by making use of results proved above. Let

$$\theta = b_1 + \frac{1}{b_2 +} \dots \frac{1}{b_m +} \frac{1}{\overset{*}{a_1} +} \frac{1}{a_2 +} \dots\dots \frac{1}{\overset{*}{a^n}}$$

$$\theta_1 = \overset{*}{a_1} + \frac{1}{a_2 +} \dots\dots \frac{1}{\overset{*}{a_n}}$$

Let $\theta_2, \theta_3 \dots \theta_n$ be defined as above. Let $\frac{p_r}{q_r}$ be the rth convergent of θ. It is easily seen that, if r is odd,

$$(\theta\, q_r^2) = q_r^2 \left| \theta - \frac{p_r}{q_r} \right|;$$

while, if r is even,

$$(\theta\, q_r^2) = 1 - q_r^2 \left| \theta - \frac{p_r}{q_r} \right|.$$

Applying these to the results established about the set

$$q_r^2 \left| \theta - \frac{p_r}{q_r} \right|$$

we get the following. I have merely enumerated the results as the verification is easy. In the following, k takes the values 1, 2 ... n:

I. n even.

(i) $(m + k)$ *odd.*

As $s \to \infty$ through 1, 2, 3, ...

$$(\theta\, q^2{}_{m+k+sn}) \to \frac{1}{\theta_k - \bar{\theta}_k}.$$

(ii) $(m + k)$ *even.*

As $s \to \infty$ through 1, 2, 3, ...

$$(\theta\, q^2{}_{m+k+sn}) \to 1 - \frac{1}{\theta_k - \bar{\theta}_k}.$$

II. n odd.

(i) $(m + k)$ *odd.*

As $s \to \infty$ through 2, 4, 6, ...

$$(\theta\, q^2{}_{m+k+sn}) \to 1 - \frac{1}{\theta_k - \bar{\theta}_k}.$$

As $s \to \infty$ through 1, 3, 5, ...

$$(\theta\, q^2{}_{m+k+sn}) \to 1 - \frac{1}{\theta_k - \bar{\theta}_k}.$$

(ii) $(m + k)$ *even.*

As $s \to \infty$ through 2, 4, 6, ...

$$(\theta\, q^2{}_{m+k+sn}) \to 1 - \frac{1}{\theta_k - \bar{\theta}_k}.$$

As $s \to \infty$ through 1, 3, 5, ...

$$(\theta\, q^2{}_{m+k+sn}) \to \frac{1}{\theta_k - \bar{\theta}_k}.$$

ON SOME RELATIONS SATISFIED BY BESSEL FUNCTIONS OF DEGREE $(n + \frac{1}{2})$

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It is easily proved that

$$P_n\left(\frac{d}{idx}\right)\frac{\sin r}{r} = i^n\sqrt{\frac{\pi}{2}}\cdot\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}}P_n(\mu),$$

where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ and $\mu = x/r$.

Consequently*

$$P_n\left(\frac{d}{idx}\right)\frac{\sin x}{x} = i^n\sqrt{\frac{\pi}{2}}\cdot\frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}}.$$

We have therefore

$$P_m\left(\frac{d}{idx}\right)\frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}} = i^{-n}\sqrt{\frac{2}{\pi}}P_m\left(\frac{d}{idx}\right)P_n\left(\frac{d}{idx}\right)\frac{\sin x}{x}$$

$$= i^{-n}\sqrt{\frac{2}{\pi}}\sum_{s=0}^{m}\frac{2n+2m-4s+1}{2n+2m-2s+1}\cdot\frac{A(n-s)\,A(m-s)\,A(s)}{A(n+m-s)}\times$$

$$P_{n+m-2s}\left(\frac{d}{idx}\right)\frac{\sin x}{x},$$

m and n being supposed to be positive integers and $m \leq n$, where †

$$A(n) = \frac{1.3.5\ldots(2n-1)}{n!}.$$

This can also be written in the form

$$P_m\left(\frac{d}{idx}\right)\frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}}$$

$$= \sum_{s=0}^{m} i^{m-2s}\,\frac{2n+2m-4s+1}{2n+2m-2s+1}\cdot\frac{A(n-s)\,A(m-s)\,A(s)}{A(n+m-s)}\times$$

$$\frac{J_{n+m-2s+\frac{1}{2}}(x)}{\sqrt{x}}.$$

* See Rayleigh, *Theory of Sound*, Art. 334, *et seq.* and Banerji, *Bull. Cal. Math. Soc*, Vol. IV, p. 8.

† See Adams, *Proc. Roy. Soc.*, 1878.

Similarly, we get

$$P_m\left(\frac{d}{idx}\right)\left\{\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}}P_n(\mu)\right\}$$

$$=\sum_{s=0}^{m} i^{m-2s}\,\frac{2n+2m-4s+1}{2n+2m-2s+1}\cdot\frac{A(n-s)\,A(m-s)\,A(s)}{A(n+m-s)}\times$$

$$\frac{J_{n+m-2s+\frac{1}{2}}(r)}{\sqrt{r}}P_{n+m-2s}(\mu).$$

For the Bessel functions of the second kind, we have

$$P_n\left(\frac{d}{idx}\right)\frac{e^{-ir}}{r}=(-1)^{n+\frac{1}{2}}\sqrt{\frac{2}{\pi}}\,\frac{K_{n+\frac{1}{2}}(r)}{\sqrt{r}}P_n(\mu),$$

and
$$P_n\left(\frac{d}{idx}\right)\frac{e^{-ix}}{x}=(-1)^{n+\frac{1}{2}}\sqrt{\frac{2}{\pi}}\,\frac{K_{n+\frac{1}{2}}(x)}{\sqrt{x}}.$$

We should therefore obtain in a similar manner,

$$P_m\left(\frac{d}{idx}\right)\frac{K_{n+\frac{1}{2}}(x)}{\sqrt{x}}$$

$$=\sum_{s=0}^{m} i^{m-2s}\,\frac{2n+2m-4s+1}{2n+2m-2s+1}\;\frac{A(n-s)\,A(m-s)\,A(s)}{A(n+m+s)}\;\frac{K_{n+m-2s+\frac{1}{2}}(x)}{\sqrt{x}}$$

and
$$P_m\left(\frac{d}{idx}\right)\left\{\frac{K_{n+\frac{1}{2}}(r)}{\sqrt{r}}P_n(\mu)\right\}$$

$$=\sum_{s=0}^{m} i^{m-2s}\,\frac{2n+2m-4s+1}{2n+2m-2s+1}\;\frac{A(n-s)\,A(m-s)\,A(s)}{A(n+m+s)}\times$$

$$\frac{K_{n+m-2s+\frac{1}{2}}(r)}{\sqrt{r}}P_{n+m-2s}(\mu),$$

n and m being supposed to be integers and $m \leq n$.

Now if we denote the operator $\frac{d}{idx}$ by D, then we can write

$$f(D)=\sum_{n=0}^{\infty}C_n P_n(D),$$

where
$$C_n=\frac{2n+1}{2}\int_{-1}^{1}f(\mu)\,P_n(\mu)\,d\mu,$$

the conditions for the validity of the expansion being the same as for a Fourier series.

It follows therefore that

$$f\left(\frac{d}{idx}\right)\frac{J_{n+\frac{1}{2}}(x)}{\surd x}=\sum_{m=0}^{\infty}C_m\,P_m\left(\frac{d}{idx}\right)\frac{J_{n+\frac{1}{2}}(x)}{\surd x}$$

$$=\sum_{m=0}^{\infty}C_m\left[\sum_{s=0}^{n\text{ or }m}i^{m-2s}\,\frac{2n+2m-4s+1}{2n+2m-2s+1}\times\right.$$

$$\left.\frac{A(n-s)\,A(m-s)\,A(s)}{A(n+m+s)}\cdot\frac{J_{n+m-2s+\frac{1}{2}}(x)}{\surd x}\right],$$

the upper limit for the summation of the series within the bracket being m or n according as $m \leq$ or $\geq n$.

Similarly,

$$f\left(\frac{d}{idx}\right)\left[\frac{J_{n+\frac{1}{2}}(r)}{\surd r}\,P_n(\mu)\right]$$

$$=\sum_{m=0}^{\infty}C_m\left[\sum_{s=0}^{n\text{ or }m}i^{m-2s}\,\frac{2n+2m-4s+1}{2n+2m-2s+1}\times\right.$$

$$\left.\frac{A(n-s)\,A(m-s)\,A(s)}{A(n+m+s)}\cdot\frac{J_{n+m-2s+\frac{1}{2}}(r)}{\surd r}P_{n+m-2s}(\mu)\right].$$

These theorems mean that if $x^{-\frac{1}{2}}J_{n+\frac{1}{2}}(x)$ or $r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(r)\,P_n(\mu)$, which is a solution of the wave equation, is differentiated in any manner the results can be expressed in a series of functions of the same type.

The same theorems are true if Bessel functions of the first kind be replaced by Bessel functions of the second kind.

Maxwell first pointed out that if a solution of the wave equation be differentiated any number of times with respect to x, y, z, we obtain solutions of the wave equation of various orders of complexities. These complex solutions cannot be interpreted physically unless they are expressed in a series of standard solutions. The method indicated above may be used with advantage to express the complex solutions in a series of standard solutions.

ON CERTAIN QUADRATIC SYSTEMS OF CONICS.

BY R. VYDYANATHASWAMY.

I. *Definition.*

If $(\lambda S + S')$ is a pair of straight lines when

$$\Delta \lambda^3 + \theta \lambda^2 + \theta' \lambda + \Delta' = 0,$$

then when $\theta' = 0$, it is known that there is an infinite number of inscribed triangles of S which are self-polar *w. r. t.* S'. In this case S will be said to be *ex-harmonic* to S', and S' *in-harmonic* to S. It will be noticed that the condition that S may be ex-harmonic to S' is linear in the co-efficients of S and quadratic in those of S'.

If S is both ex- and in-harmonic to S', S will be said to be harmonic to S'.

The inscribed triangles of S belonging to a given pencil are all self-polar triangles of one and only one conic S'*. Thus there may be established a correspondence between pencils of cubics and the in-harmonic conics of S. In this correspondence the harmonic conics of S will correspond to the Null pencils †.

II. A (1, 1) correspondence may be established between the points of a five-dimensional space S_5 and all conics in a plane; for instance between the point in S_5 whose homogeneous co-ordinates are $(a\, b\, c\, f\, g\, h)$ and the plane conic whose equation is $(a\, b\, c\, f\, g\, h)\ (x, y, z)^2 = 0$. By this correspondence the straight lines of S_5 will correspond to four-point systems of conics, and the planes of S_5 will correspond to nets of conics, *i.e.*, systems of the type $\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3$ where S_1, S_2, S_3 are three conics. The conics which are line-pairs will correspond to points in S_5 lying on the cubic surface

$$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

A quadric surface in S_5 will correspond to a general quadratic system of conics, namely the set of conics whose co-efficients satisfy a general quadratic relation. Now a quadric surface in S_5 has always two kinds of generating planes, such that any two generating planes of the same kind intersect in one and only one point, and two generating planes of different kinds either do not intersect at all or intersect in a straight line ‡. Hence

* *Vide* 'Linear Systems of the Third Order on the Conic,' *J. I. M. S.*, August 1921.

† *Ibid.*

‡ 'Quadric in Five Dimensions,' *J. I. M. S.*, June 1920).

A quadratic system of conics necessarily contains ∞^3 *nets. These nets can be divided into two distinct classes, such that two nets of the same class have one and only one conic in common, and two nets belonging to different classes have either no conic or a four-point system of conics in common.*

This theorem is fundamental in the study of quadratic systems of conics. We propose in this note to study the application of this theorem to a particular type of quadratic systems, namely, those systems which correspond in S_5 to the polar quadrics of the cubic surface Δ. These polar quadrics have equations of the form

$$a' (bc - f^2) + b' (ca - g^2) + c' (ab - h^2) + 2f' (gh - af) + 2 g' (hf - bg) + 2h' (fg - ch) = 0$$

where $(a, b...)$ are current co-ordinates and $(a', b'...)$ constants. Now this equation simply expresses that the conic $(a\ b\ c\ f\ g\ h)$ is in-harmonic to a fixed conic $(a'\ b'\ c'\ f'\ g'\ h')$. Thus the polar quadrics of Δ correspond to special quadratic systems, namely, systems consisting of all conics in-harmonic to a fixed conic.

III. Consider now the family F of conics in-harmonic to a given conic S. Let s be the point in S_5 corresponding to S and let Σ be the polar quadric of s *w.r.t.* Δ; then Σ is the locus in S_5 corresponding to the family F. Further if the polar plane that is four-dimensional flat region) of S *w. r. t.* Σ intersect the latter in the locus Σ_1, then clearly Σ_1 is the locus of points which correspond to harmonic conics of S.

Now s determines an involutoric correspondence I_s between the points of Σ, namely, the correspondence of the two intersections of Σ with any line through s. The self-corresponding points of I_s are clearly the points on Σ_1. Further it is not difficult to shew that the two *kinds* of generating regions of Σ are interchanged by I *.

The meaning of the transformation I_s in the plane is evident. Thus if S′ is an in-harmonic conic of S, $I_s S'$ will be the *other* in-harmonic conic of S contained in the four-point system (S, S'). Also, from what has preceded, this correspondence interchanges the two classes of in-harmonic nets of S—that is the nets which belong to the family of conics in-harmonic to S.

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* This is easily seen by the method of correspondence adopted in 'Quadric in Five dimensions.' In this method s will correspond to a system of forces and points on Σ to single forces. The correspondence I_s will then be the correspondence of conjugate forces *w.r.t.* the system s. This latter correspondence will obviously replace each point by a plane, *viz.*, its nul-plane and *vice versa.* Since points and planes correspond to the two kinds of generating regions of Σ, it follows that I_s interchanges the two kinds of generating regions.

Hence we have

The conic S determines an involutoric correspondence between its in-harmonic conics—the self-corresponding conics of which are its harmonic conics. This correspondence interchanges the two classes of the in-harmonic nets of S.

The operation I will be called *inversion*, and corresponding in-harmonic conics of S will be called *inverts* of each other in S.

Ex. (*1*). The invert of an in-harmonic conic having double contact with S is a repeated line, *viz.*, the chord of contact. *Vice versa*, any repeated line is an in-harmonic conic of S and its invert is a conic having double contact with S.

Ex. (*2*). The pencils of inscribed Δs determined on S by any in-harmonic conic and its invert are mutually harmonic pencils (*i.e.* any triad of either pencil is harmonic to any triad of the other).

IV. *The in-harmonic nets of S.*

If ABC is any inscribed Δ of S, then, clearly, the net of conics having ABC for a self-polar triangle is an in-harmonic net of S. Let us term this, the in-harmonic net of the first class corresponding to the inscribed triangle ABC. This net contains three repeated lines, *viz.*, the sides of ABC and, further, is easily seen to contain only one conic having double contact with S—the chord of contact being the polar line of the triangle ABC *w.r.t.* S.

The invert of this net we call the in-harmonic net of the second class corresponding to the triangle ABC. In view of Ex. (1) III, this net should contain *one* repeated line, namely, the polar line of ABC and *three* conics having double contact with S at the corners of ABC.

A four-point system every conic of which is harmonic to S, may itself be said to be harmonic to S. Clearly such a four-point system is inverted into itself by S. It is easy to see that any net of the first class determined by an inscribed triangle ABC can contain only one four-point system harmonic to S; for there is only one quadrangle which has ABC for harmonic triangle and which is in addition self-polar *w.r.t.* S. By inversion, it follows that this same harmonic system is contained in the net of the second class determined by ABC. Hence the net of the second class may be conceived as the set of conics having double contact with the conics of a four-point system L, harmonic to S, the chord of contact being always a particular line, namely, the polar line of the harmonic triangle of the quadrangle which determines L.

1. Finally we may verify the theorem, concerning the common conics of two nets. Clearly any two nets of the first class, say those determined

by the triangles ABC, A′B′C′, have one and only one conic in common, namely, the unique conic which has ABC and A′B′C′ for self-polar triangles. By inversion it follows that any two nets of the second class have one and only conic in common. Consider now two nets of different classes, say a net N_1 of the first class determined by ABC and a net N_2 of the second class determined by A′B′C′. The common conics of these two nets must clearly consist of those conics of N_1 whose inverts belong to the net N of the first class determined by A′B′C′. Hence, in view of Ex. (2) III above, N_1 and N_2 have no common conics unless ABC and A′B′C′ are harmonic. When these triangles are harmonic, we may easily shew that the common conics are all the conics of the four-point system of N_1, for which (h_1, h_2) is a conjugate pair; h_1, h_2 being the points of intersection with S of the polar line of A′B′C′. For, the pencils determined on S by the conics of this four-point system are all such that their harmonic pencils (Ex. (2) quoted above) contain ABC. Hence the inverts of the conics of this system belong to N. Thus nets of different classes have either no conic in common or a four-point system in common.

Ex. (1). If ABC, A′B′C′ are harmonic inscribed triangles of S and if their polar lines cut S in (h_1, h_2), (h_1', h_2') respectively, shew that the invert of a conic which has ABC for self-polar triangle and h_1', h_2' for conjugate points, is a conic which has A′B′C′ for self-polar triangle and h_1, h_2 for conjugate points.

Ex. (2). Shew that any in-harmonic four-point system of S must have its common self-polar triangle inscribed in S. Shew that any in-harmonic conic of S belongs to ∞ in-harmonic four-point systems.

THE PEDAL LINE FAMILY OF A TRIANGLE

By A. Narasinga Rao.

1. *Introductory.*—In this paper an attempt is made to study the distribution and properties of the pedal lines of a triangle, regarded not individually but as members of a linear aggregate. Though the results obtained are mostly well known, it is hoped that the symmetry and simplicity of the methods employed will be of interest. The problem of determining all triangles having a common pedal line system is also taken up for solution at the end of the paper.

2. *The Pedal Line.*—Let A, B, C be a triangle and L a point on its circumcircle. It is well-known that the feet of the perpendiculars from L on the sides of ABC lie on a line called the Simpson or the pedal line of L *w. r. t.* the triangle. As L moves round the circle, we obtain a singly infinite system of pedal lines which could be associated with the continuous variations of a single parameter l. We shall denote the vectorial angle of any pt. L on the circle by the corresponding Greek letter λ and $\tan \frac{\lambda}{2}$ by l so that as l varies from $-\infty$ to $+\infty$ the pt. L moves once round the circle. Thus the parameters of the vertices A, B, C are a, b, c and those of the circular points I and J are $+i$ and $-i$.

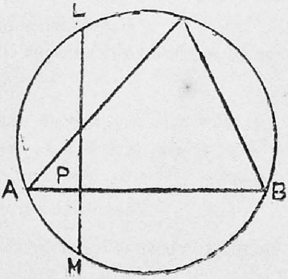
3. *Some particular cases.*—From the definition of the pedal line of a point it follows that :—

(1) The pedal lines of the vertices of the triangle are the perpendiculars from them to the opposite sides.

(2) The sides of the triangle ABC are themselves the pedal lines of the points diametrically opposit to them on the circumcircle.

(3) The pedal line of the points I J is the line at infinity. For if the pedal of I cuts AB in P then I must lie on the line through P $\perp$ AB and this will not be true unless P be at infinity. Similar considerations apply to the other two sides BC, CA. In the same manner the pedal of J is also the line at infinity.

4. *Class of the Envelope.*—Take a pt. P on AB and let LPM be a chord $\perp$ to AB. Then through the point P there pass three pedal lines, the line AB, the pedal line of L and that of M and it is easy to see that no other pedal line can possibly pass through it. We may infer by the principle of continuity, that three pedal lines pass through any given pt. and that their linear aggregate envelope a curve of the



third class of which the line at infinity is a bitangent as it corresponds to two distinct values $\pm i$ of the parameter.

5. *Concurrent Pedals.*—Let us now determine the relation connecting the parameters l, m, n of 3 concurrent pedals. The relation is obviously linear and symmetrical, since given 2 pedal lines, there is a unique third passing through their common point. Let us assume it to be

$$A\, lmn + B\,(lm + mn + nl) + C\,(l + m + n) + D = 0 \ldots \qquad \text{I.}$$

or slightly $l\,[A\, mn + B\,(m + n) + C] + [B\, mn + C(m + n) + D] = 0.$

The later form shows that such a lineo-linear system contains always a singular pair m_1, n_1 (obtained by solving simultaneously $Amn + Bm+n + C = 0$ and $Bmn + Cm+n + D = 0$) which form a triad with any l. In this case the singular pair is obviously $(+i, -i)$ since the pedals of these points, and any other point are all concurrent.

Hence we have $\qquad A + C = 0, \;\; B + D = 0 \qquad \ldots \qquad \ldots \quad$ II

Also a, b, c form a triad satisfying I since the corresponding lines (the perpendiculars of the triangle) are concurrent.

We thus obtain the ratios of A, B, C, D and the condition for concurrence is finally found to be

$$\frac{\Sigma l - lmn}{1 - \Sigma lm} = -\frac{B}{A} = \frac{\Sigma a - abc}{1 - \Sigma ab} \qquad \ldots \qquad \ldots \quad \text{III}$$

or $\qquad$ remembering that $l = \tan\frac{\lambda}{2}$ $\;\; a = \tan\frac{\alpha}{2}$, etc.

we have $\qquad \tan\dfrac{\lambda + \mu + \nu}{2} = \tan\dfrac{\alpha + \beta + \gamma}{2}$

i.e., $$\lambda + \mu + \nu = \alpha + \beta + \gamma + 2k\pi \ldots \qquad \ldots \quad \text{IV}$$

as the condition for the concurrency, a result which does not seem to be generally known.

Cor 1. If the pedal lines of L, M, N with respect to the triangle ABC are concurrent, so also are the pedals of A, B and C *w. r. t.* the $\triangle$ LMN.

Cor. 2. By a proper choice of origin, the condition of concurrency can be reduced to the simple form

$$lmn = l + m + n.$$

(For this we choose the origin of angular measurement so that $\alpha + \beta + \gamma = 0$ or $2k\pi$.) We shall hereafter take this to be the case always.

Cor. 3. The pedal corresponding to the parameter l touches the envelope where it is cut by that of $\dfrac{2l}{l^2 - 1}$.

Cor. 4. The pedal lines corresponding to the points $l^3 = 3l$, *i.e.*, $l = 0 \pm \sqrt{3}$ are the cuspidal tangents of the envelope. These tangents are themselves concurrent.

6. *The Equation of the Pedal Line in areal co-ordinates.*—The equation of the pedal line of the pt. θ *w. r. t.* the triangle $\alpha\beta\gamma$ is of the form

$$x\,\mathrm{L} + y\,\mathrm{M} + z\,\mathrm{N} = o$$

where L M and N are cubics in $t \left(= \tan \frac{\theta}{2}\right)$.

Now when $t = -\frac{1}{a}, -\frac{1}{b}$ or $-\frac{1}{c}$ the corresponding lines are $x=0$, $y = 0$ or $z = 0$. Hence the equation can now be written as

$$x\,(tb + 1)\;(tc + 1)\;\mathrm{L}' + y\;(ta + 1)\;(tc + 1)\;\mathrm{M}' + z\ldots\ldots\ldots = 0$$

where L′ M′ and N′ are linear in t.

Again when $t = a$ the corresponding pedal is

$$\frac{y}{z} = \frac{\tan \mathrm{B}}{\tan \mathrm{C}} \text{ or } \frac{y\,(a - b)}{1 + ab} + \frac{z\,(a - c)}{1 + ac} = 0$$

and similarly for $t = b$ and c.

Hence $\mathrm{L}' = (t - a)\,\mathrm{L}''$, $\mathrm{M}' = (t - b)\,\mathrm{M}''$, $\mathrm{N}' = (t - c)\,\mathrm{N}''$ where L″, M″ and N″ are constants. Finally applying the condition that the pedal of $t = i$ is the line $x + y + z = o$, we have $\mathrm{L}'' = \mathrm{M}'' = \mathrm{N}''$ giving us the required equation

$$\frac{x\,(t - a)}{1 + ta} + \frac{y\,(t - b)}{1 + tb} + \frac{z\,(t - c)}{1 + tc} = 0$$

$$\text{or } x \tan \frac{\theta - \alpha}{2} + y \tan \frac{\theta - \beta}{2} + z \tan \frac{\theta - \gamma}{2} = 0.$$

7. *Perpendicular pedal lines.*—As an example of the methods employed in this paper, let us determine the relation between the parameters of perpendicular pedals. To each l there corresponds a single direction, a definite perp. direction and 3 lines along this direction. But as the line at infinity counts twice as a pedal in this perp. direction there is only one other distinct line corresponding, say, to the parameter m. The relation connecting l and m is thus a one-to-one correspondence of the type

$$\mathrm{A}\,lm + \mathrm{B}\,(l + m) + \mathrm{C} = 0.$$

But we know that the pedals corresponding to the parameters $\left(a, -\frac{1}{a}\right)$ $\left(b, -\frac{1}{b}\right)$ $\left(c, -\frac{1}{b}\right)$ are at rt. angles (being in fact the sides and the perps. of the Δ). Hence these pairs satisfy the relation which must therefore be

$$lm + 1 = 0 \quad \text{or } \tan \frac{\lambda}{2} \tan \frac{\mu}{2} + 1 = 0.$$

showing that the pedals of the extremities of a diameter meet at right angles. It is a well-known theorem that they meet at right angles on the nine points circle.

8. *The Envelope of the Family.*—We have seen already that the line at infinity (L_∞) is a bitangent of the envelope. Also since 3 tangents pass through any given point, and as no other pedal can pass through I other than L_∞, I must be one of the points of contact and similarly J. The envelope is thus seen to be a curve of the 3rd class having double contact with L_∞ at the circular points, *i.e.*, a tricusp Hypocycloid. The intersections of ⊥ tangents lie on the in-circle of the Hypocycloid. This is the nine-points circle of the triangle. The cusp lies on a concentric circle of thrice its radius. The equations of the cuspidal tangents have been already obtained. These cuspidal tangents are equally inclined to one another and meet at the nine-points circle of ABC.

9. Are there other triangles having the same pedal line family as the △ ABC ? The question is easily answered in the affirmative. Any transformation which transforms ⊥ lines into ⊥ lines and which leaves the hypocycloid unchanged conserves the pedal line group and such a transformation may give us a new triangle. Rotation through $\pm 120_0$ about the nine-point centre and reflexion about the 3 cuspidal tangents readily suggest themselves. More generally the hypocycloid is completely determined when we are given its in-circle and one of its cuspidal tangents both in position and magnitude. Hence if we are given a circle and one of its diameters, all triangles constructed so as to have this circle for their nine-points circle and the given diameter for a pedal line will have a common pedal line group, namely, the tangents of the unique hypocycloid escribed to this circle and having the diameter for a cuspidal tangent. There is an infinite number of triangles satisfying the condition.

SOLUTIONS.

Question 417.

(V. Ramaswami Aiyar, M.A.) :—If parabolas are escribed to a triangle, shew that any system of corresponding lines connected with them (considered as similar figures) envelope a three-cusped hypocycloid.

Question 1101.

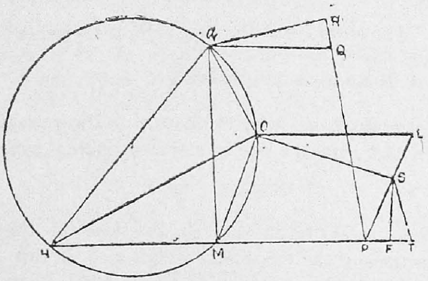
(M. Bhimasena Rao):—Show that the centre of a three-cusped hypocycloid escribed to a given triangle is equidistant from the orthocentre and the circum-centre of the triangle.

Question 1250.

(V. Ramaswami Aiyar, M.A.):—If five straight lines be tangents to a three-cusped hypocycloid, prove that the foci of the five parabolas touching the lines taken four at a time are all collinear.

Solution and remarks by M. Bhimasena Rao.

Let S be the focus and F the foot of the directrix of a parabola escribed to a triangle ABC whose ortho-centre and circum-centre are H and O respectively, and let PQR be any line meeting the directrix FH in P. If ST be drawn parallel to PQR, meeting the directrix in T, we have to prove that PQR touches a three-cusped hypocycloid when the triangle STP is given in species.



On HO describe a triangle HON inversely similar to STP, and draw the circle HON, meeting the directrix in M. Join MN.

$$\hat{SPT} = \hat{HNO} = \hat{OMP} = \theta, \text{ say.}$$

$\therefore$ SP is parallel to OM.

Again $\hat{STP} = \hat{HON} = \hat{NMH} = \phi$, say.

$\therefore$ ST is parallel to NM.

$\therefore$ NM is parallel to PQR.

Complete the parallelograms OMPL and NMPQ.

Considering the triangle OSL, we have

$$\frac{OL}{OS} = \frac{\sin\ OSL}{\sin\ OLS} = \frac{\sin(OSF - PSF)}{\sin\ SPT}$$

Now, OS = R, the circum-radius of ABC, angle OSF $= \lambda + \mu + \upsilon$ (Gallatly's Modern Geometry, page 27), if the trilinear co-ordinates of S referred to ABC be sec λ, sec μ, sec υ. The angle SPT $= \theta$ and PSF is $90 - \theta$.

$$\therefore OL = -R\cos(\lambda + \mu + \upsilon + \theta)/\sin\theta.$$

If p is the length of the perpendicular NR on PQR, we have

$$p = NR = NQ \sin NQR = MP \sin MPR$$

$$= OL \sin\phi$$

$$= -R\sin\phi\cos(\lambda + \mu + \upsilon + \theta)/\sin\theta.$$

In terms of a single variable ω, $-\lambda$, μ, υ may be expressed as $180^\circ + \omega$, $\omega - B$, and $\omega + C$ respectively.

The envelope of the line PQR is therefore

$$p = R\sin\phi\cos(3\omega - B + C + \theta)/\sin\theta.$$

which is the tangential polar equation of a three-cusped hypocycloid whose centre is at N, and the raduis of whose rolling circle is

$$R\sin\phi/\sin\theta.$$

If PQR is a tangent to the parabola, the sides of the triangle ABC will be the positions of the variable line PQR, and in this case the three-cusped hypocycloid is inscribed in the triangle ABC. Since SP will now bisect the angle QPT, the triangle SPT is isosceles, SP being equal to PT; and therefore in the similar triangle HON, HN = ON. This is the result of Question 1101.

Next consider a triangle ABC and two transversals XYZ and X′Y′Z′. If S and S′ are the foci of the parabolas escribed to ABC and touching

XYZ′ and X′Y′Z respectively, SX, SY, SZ are equally inclined to the sides of ABC, say, at an angle θ, and likewise S′X′, S′Y′ and S′Z′ at an angle ϕ. These lines being concurrent corresponding lines of directly similar figures described on XX′, YY′, ZZ′, the points of concurrence, namely, S and S′, and the double points of the similar figures, being the foci of the remaining three parabolas touching the two transversals and two of the three sides of ABC, lie on the circle of similitude. This circle is the well-known *Miquel's* circle of the lines AB, BC, CA, XYZ and X′Y′Z′. If a three-cusped hypocycloid touches these five lines, then XYZ and X′Y′Z′ are corresponding lines of the parabolas having S and S′ as foci; the angle $\theta = \phi$, that is to say, SX, SY, SZ are respectively parallel to S′X′, S′Y′, S′Z′. The circle of similitude, on account of this parallelism breaks up into the line SS′ and the line at infinity. If S_1, S_2, S_3 are the double points of the similar figures on XX′, YY′, ZZ′, then AS_1, BS_2, CS_3 will now be parallel and inclined to SS′ at the same angle θ, and the invariable points are at infinity on lines inclined to the sides of ABC at the angle θ.

The tangential equation of the three-cusped hypocycloid in Question 417 may be obtained thus :—

If λ, μ, ν be the direction angles of any line, the equation of the vertex-tangent of the parabola escribed to ABC and having its focus at

$$a \sec \lambda,\ b \sec \mu,\ c \sec \nu \text{ is, in areals,}$$

$$L \equiv \alpha \tan \mu \tan \nu + \beta \tan \nu \tan \alpha + \gamma \tan \alpha \tan \beta = 0.$$

The axis of the parabola is

$$M \equiv \alpha (\tan \mu + \tan \nu) + \beta (\tan \nu + \tan \alpha) + \gamma (\tan \alpha + \tan \beta) = 0.$$

Also let $N \equiv \alpha + \beta + \gamma = 0$, the line at infinity.

Since the vertex-tangents of escribed parabolas are corresponding lines, and likewise their axes, the line at infinity being a self-corresponding line, the lines

$$l L + m M + n N = 0$$

where l, m, n are constants, are corresponding lines of the parabolas for variable values of λ, μ, ν.

The tangential equation of the envelope of these lines is obtained from eliminating λ, μ, ν, ρ from the following equations :—

$$l \tan \mu \tan \nu + m (\tan \mu + \tan \nu) + n = \rho x$$

$$l \tan \nu \tan \lambda + m (\tan \nu + \tan \lambda) + n = \rho y$$

$$l \tan \lambda \tan \mu + m (\tan \lambda + \tan \mu) + n = \rho z$$

$$a \cos \lambda + b \cos \mu + c \cos \nu = 0$$

$$a \sin \lambda + b \sin \mu + c \sin \nu = 0$$

The steps in the elimination are omitted as being a little tedious, and writing $y - z = \xi$, $z - x = \eta$ and $x - y = \zeta$, the eliminant may be expressed, by geometrical considerations, in the form

$$l \Sigma \cot A \, \xi^2 x + m \, \xi \eta \zeta = \frac{nl - m^2}{(l^2 + m^2)^2} \pi \left\{ l \, (\eta \cot \beta - \zeta \cot c) - m\xi \right\}$$

The envelope of the vertex-tangent L = 0, is the Steiner's tri-cusp

$$\Sigma \cot A \, \xi^2 \, x = 0.$$

The envelope of the axis M = o is the Steiner's tri-cusp (of the anti-medial triangle of ABC)

$$\Sigma \cot A \, \xi^2 \, (-x + y + z) = 0.$$

The condition that the line $l\,L + m\,M + n\,N = 0$ touches the parabola is

$$nl - m^2 = 0,$$

and the envelope of the line will then be

$$l \, \Sigma \cot A \, \xi^2 x + m \, \xi \eta \zeta = 0,$$

a three-cusped hypocycloid inscribed in ABC, and the centre of this curve is $l \, \Sigma \, x \sin A \cos (B - c) - m \, \Sigma \, x \sin A \sin (B - c) = 0$, the locus of the centre is the line $\left(\frac{\sin 3A}{\sin A}, \frac{\sin 3B}{\sin B}, \frac{\sin 3C}{\sin C} \right)$ which is the perpendicular bisector of OH.

Question 1011.

(SELECTED) ;—In an ellipse the tangent at P cuts the directrices in Z, Z′, and the remaining tangents from Z, Z′ to the ellipse meet at T. Show that PT is normal to the ellipse and bisected by the minor axis.

Solution by R. P. Paranjpye and K. J. Sanjana.

The points of contact of the tangents ZT and Z′T are the other extremities of the focal chords through P.

Considering the triangle PQS′, QT bisects the exterior angle at Q and S′T the exterior angle at S′. Therefore T is the ex-centre of the triangle and PT bisects the angle QPS′ and is therefore the normal at P.

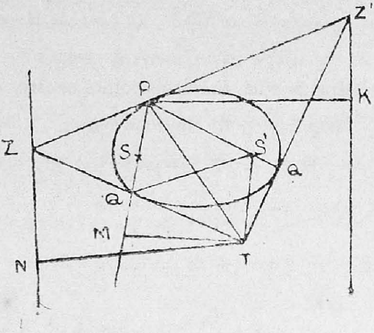
Draw TM perpendicular to PS and TN and PK perpendicular to the directrices ; PM is equal to half the perimeter of the triangle PQS′ and, therefore, PM is equal to the major axis.

$$\therefore \quad SM = PS'.$$

$$\therefore\ TN/e = PK/e.$$

$$\therefore \quad TN = PK.$$

$\therefore$ PT is bisected by the minor axis.



Question 1061.

(LAKSHMISANKER N. BHATT) :—The Euler line of the triangle AB_1C_1 meets the sides AB_1, AC_1 in B_2, C_2 ; The Euler line of AB_2C_2 meets these sides in B_3, C_3 ; and this process in continued indefinitely. If N_r is the nine-point centre of the r^{th} triangle AB_rC_r thus formed, and AN_r meets B_rC_r in D_r, prove that the straight lines $N_x\ N_{x+2y}$ and $D_x\ D_{x+2y}$ are parallel, x and y being any positive integers.

Solution by K. Satyanarayana.

By Question 1009 solved on page 185 of the *J. I. M. S.*, Vol. XI, it follows that

$B_x C_x$, $B_{x+2}C_{x+2}$, $B_{x+4}C_{x+4}$, ... are parallel. Hence $\triangle$s AB_xC_x and $AB_{x+2y}C_{x+2y}$ are similar and N^x, N_{x+2y}, D_x, D_{x+2y} are corresponding points

$$\therefore \frac{AN_x}{AD_x} = \frac{AN_{x+2y}}{AD_{x+2y}}$$

$\therefore$ N_xN_{x+2y} is parallel to D_xD_{x+2y}.

Question 1100.

(Lakshmishanker N. Bhatt):—ABC is a triangle having its sides divided internally and externally in the ratios p/q, q/r, r/p. If the four lines joining the points of division, three and three, touch a parabola, then prove that

(i) the parabola can for no values of p/q, q/r, r/p pass through the intersections of the circum-circle and the nine-points circle;

and (ii) if p, q, r are proportional to

$$\sqrt{\{a(\cos A - \cos B \cos C)\}},\ \sqrt{\{b(\cos B - \cos A \cos C)\}},$$
$$\sqrt{\{c(\cos C - \cos A \cos B)\}}$$

then the focus will coincide with the nine-points centre.

Solution by K. Satyanarayana.

Let D, D′; E, E′; F, F′ divide respectively BC, CA, AB internally and externally in ratios $\frac{q}{r}, \frac{r}{p}, \frac{p}{q}$.

Co-ordinates of D, E, F *w. r. t.* $\triangle$ ABC are

$$0, \frac{r}{q+r}, \frac{q}{q+r};\ \frac{r}{r+p}, 0, \frac{p}{r+p};\ \frac{q}{p+q}, \frac{p}{p+q}, 0.$$

Lines EFD′, FDE′, DEF′, D′E′F′ are $-px+qy+rz=0$;

$px-qy+rz=0$; $px+qy-rz=0$; $px+qy+rz=0$.

If co-ordinates of P *w. r. t.* $\triangle$ ABC, and $\triangle$ DEF be x, y, z; X, Y, Z,

$$X = \frac{\triangle PEF}{\triangle DEF} = \begin{vmatrix} x & y & z \\ \frac{r}{r+p} & 0 & \frac{p}{r+p} \\ \frac{q}{p+q} & \frac{p}{p+q} & 0 \end{vmatrix} \div \begin{vmatrix} 0 & \frac{r}{q+r} & \frac{q}{q+r} \\ \frac{r}{r+p} & 0 & \frac{p}{r+p} \\ \frac{q}{p+q} & \frac{p}{p+q} & 0 \end{vmatrix}$$

$$= \frac{p(q+r)(-px+qy+rz)}{2pqr}$$

$$\therefore \quad \frac{\dfrac{X}{p(q+r)}}{-px+qy+rz} = \frac{\dfrac{Y}{q(r+p)}}{px-qy+rz} = \frac{\dfrac{Z}{r(p+q)}}{px+qy-rz}$$

$$= \frac{\dfrac{X}{p(q+r)} + \dfrac{Y}{q(r+p)} + \dfrac{Z}{r(p+q)}}{px+qy+rz}.$$

$\therefore$ the parabola is $\sqrt{\lambda X} + \sqrt{\mu Y} \sqrt{\upsilon Z} = 0$; where $\lambda + \mu + \upsilon = 0$.

But D′ E′ F′ is $\dfrac{X}{p(q+r)} + \dfrac{Y}{q(r+p)} + \dfrac{Z}{r(p+q)} = 0,$

and because this touches the parabola, we get

$$\lambda p(q+r) + \mu q(r+p) + \upsilon r(p+q) = 0.$$

Hence $\dfrac{\lambda}{p(q-r)} = \dfrac{\mu}{q(r-p)} = \dfrac{\upsilon}{r(p-q)}$

$\therefore$ the parabola is $\sqrt{p^2(q^2-r^2)(-px+qy+rz)}$

$$+ \sqrt{q^2(r^2-p^2)(px-qy+rz)} + \sqrt{r^2(p^2-q^2)(px+qy-rz)} = 0.$$

which reduces to $\dfrac{x^2}{q^2-r^2} + \dfrac{y^2}{r^2-p^2} + \dfrac{z^2}{p^2-q^2} = 0.$

(i) If this should be $\sum a^2 yz + k[\sum(b^2+c^2-a^2)x^2 - 2\sum a^2yz]$

$= 0$, then $k = \frac{1}{2}$ and

$$\frac{q^2-r^2}{(c^2+a^2-b^2)(a^2+b^2-c^2)} = \frac{r^2-p^2}{(a^2+b^2-c^2)(b^2+c^2-a^2)}$$

$= \dfrac{p^2-q^2}{(b^2+c^2-a^2)(c^2+a^2-b^2)}$ which cannot hold, since sum of denominators $-[\sum a^4 - 2\sum b^2c^2]$ or $(a+b+c)(-a+b+c)\times(a-b+c)(a+b-c)$ can never be zero; $\because$ sum of any two of the quantities a, b, c is always greater than the third.

(ii) In this case equation of parabola becomes

$$\frac{x^2}{(b^2-c^2)(b^2+c^2-a^2)} + \frac{y^2}{(c^2-a^2)(c^2+a^2-b^2)} +$$

$$\frac{z^2}{(a^2-b^2)(a^2+b^2-c^2)} = 0.$$

By Example 7, page 310 of Askwith's Analytical Geometry,

the co-ordinates of the focus are proportional to $\dfrac{b^2}{(c^2-a^2)(c^2+a^2-b^2)}$

$$+ \frac{c^2}{(a^2-b^2)(a^2+b^2-c^2)}, \text{ etc.,}$$

i.e., to $\dfrac{(b^2-c^2)(a^4-b^4-c^4)}{(c^2-a^2)(a^2-b^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}$, etc.,

i. e., to $(b^2-c^2)^2(b^4+c^4-a^4)(b^2+c^2-a^2)$, etc.

But the co-ordinates of N. P. centre are proportional, $a^2 (b^2 + c^2) - (b + c^2)^2$, etc., which are not proportional to those of the focus.

The second portion of the problem therefore appears to be incorrect.

Question 1122.

(V. Ramaswami Aiyar) :—If the join of four concyclic points A, B, C, D taken in pairs intersect in P, Q, R, prove that the N.P. circle of PQR passes through the centroid of ABCD.

Solution by K. Satyanarayana.

O is the centre of the circle ABCD.

X, Y, Z, W, L, M, N are the mid.-points of AB, BC, CD, DA, QR, RP, PQ, respectively, G being the centroid.

From quadrilaterals RCPD, and QCPB, respectively, it follows that MZX, NYW are straight lines.

Also, from the cyclic quadrilaterals,

$$\angle XZQ = \angle GZC = \angle QOX,$$

$$\angle WYR = \angle GYC = \angle ROW,$$

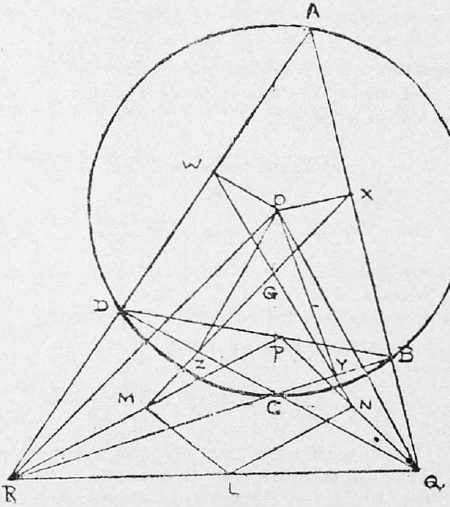
$$\angle DCB = \angle ZCY = \angle WOX;$$

$$\therefore \quad \angle ZGY = \angle MGN = \angle ROQ$$

But $\angle$ROQ is the supplement of $\angle$ RPQ or $\angle$MLN since O is the ortho-centre for Δ PQR.

Hence $\angle$MGN + $\angle$MLN = 2 right angles.

Thus G lies on the circle LMN, which is the N.P. circle of PQR.



Question 1163.

(V. Ramaswami Aiyar) :—If a rectangular hyperbola passes through the in-centre of a triangle and the feet of the perpendiculars drawn therefrom to the sides, prove that it cuts the inscribed circle again at the point which is diametrically opposite the Feuerbach point.

Solution (1) by G. A. Srinivasan.

Reciprocate with respect to the in-circle. The rectangular hyperbola reciprocates into a parabola which touches the sides of the given triangle and whose directrix passes through the in-centre. We have only to prove that the fourth common tangent of this parabola and the in-circle is parallel to the Feuerbach tangent.

Let the equation to the parabola (in areals) be

$$\sqrt{\lambda x} + \sqrt{\mu y} + \sqrt{\nu z} = 0 \quad \ldots \quad \ldots \quad (1)$$

where

$$\lambda + \mu + \nu = 0 \quad \ldots \quad \ldots \quad (2)$$

Its directrix is *

$$\lambda\,(b^2 + c^2 - a^2)\,x + \mu\,(c^2 + a^2 - b^2)y + \nu\,(a^2 + b^2 - c^2)\,z = 0.$$

Since this passes through the in-centre (a, b, c)

$$\lambda \cos A + \mu \cos B + \nu \cos C = 0. \quad \ldots \quad (3)$$

From (2) and (3), we obtain

$$\frac{\lambda}{(b-c)(s-a)} = \frac{\mu}{(c-a)(s-b)} = \frac{\nu}{(a-b)(s-c)}$$

Let the fourth common tangent of this parabola and the in-circle be $lx + my + nz = 0 \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$

$$\text{Then} \quad \frac{s-a}{l} + \frac{s-b}{m} + \frac{s-c}{n} = 0$$

$$\text{and} \quad \frac{(s-a)(b-c)}{l} + \frac{(s-b)(c-a)}{m} + \frac{(s-c)(a-b)}{n} = 0,$$

$$\text{whence,} \quad l = \frac{s-a}{b+c-2a},\ m = \frac{s-b}{c+a-2b},\ n = \frac{s-c}{a+b-2c}.$$

Substituting in (4) we obtain

$$\frac{(s-a)}{b+c-2a}\,x + \frac{(s-b)}{c+a-2b}\,y + \frac{s-c}{a+b-2c}\,z = 0.$$

---

* [Vide Ex. **14**, p. **312**, Askwith's Analytical Geom. of the Conic Section (old edition)]

This line is evidently parallel to the Feuerbach tangent of the in-circle given by †

$$\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0.$$

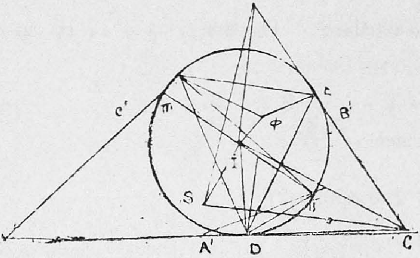
Remarks.—

The equation to the rectangular hyperbola in question is easily seen to be

$$\Sigma\,(b-c)\,(s-a)^2x^2+\Sigma\,(b-c)\,(s-b)\,(s-c)\,yz=0.$$

This conic passes through the point whose co-ordinates are $\frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}$, *i.e.* the Gergonne point or the point of concurrence of the lines joining the vertices to the points of contact of the in-circle with the opposite sides.

Solution (2) *by S. L. Malurkar.*



Reference :—I in-centre of △ ABC.

△ DEF pedal △ of II.

ϕ the ortho-centre of △ DEF.

S the circum-centre of △ ABC.

π The Feuerbach point for the in-⊙ of △ ABC.

π' The diametrically opposite point.

A′ B′, C′, the middle points of BC, CA and AB.

We know π' (DEFI)

$$=\frac{\sin D\pi'E\,.\,\sin F\pi'I}{\sin D\pi'I\,.\,\sin F\pi'E}$$

$$=\frac{\sin DFE}{\sin D\pi'\pi}\,.\,\frac{\sin F\pi'\pi}{\sin FDE}$$

$$=\frac{\sin DFE}{\sin FDE}\,.\,\frac{\sin \pi FA}{\sin \pi DC}$$ $\because \angle\pi\,FA = \angle$ in alternate segment $\angle F\,\pi'\,\pi$.

$$=\frac{\sin DFE}{\sin FDE}\,.\,\frac{\sin \pi FC'}{\sin \pi DA'}$$

† [Vide Askwith, *l. c.* p. 308].

$= \frac{\sin DFE}{\sin FDE} \cdot \frac{\sin CIS}{\sin AIS}$ $\because \triangle A'\pi D$ is similar to $\triangle$ SAI and $\triangle C'\pi F$ .... $\triangle$ SCI.

$= \frac{\sin DFE}{\sin FDE} \cdot \frac{\sin SCI}{\sin SAI}$ $\because$ AS = CS and IS common in the $\triangle$ SIA and $\triangle$ SIC.

$= \frac{\sin DFE}{\sin FDE} \cdot \frac{\sin IF\phi}{\sin ID\phi}$ $\because \angle ID\phi = \frac{1}{2}\{D\hat{F}E - D\hat{E}F\}$

$= \frac{1}{2}[\angle C - \angle B] = S\hat{A}I.$

$= \frac{\sin D\phi E}{\sin F\phi E} \cdot \frac{\sin I\phi F}{\sin I\phi D}$. $\because$ IF = ID and $I\phi$ common in the triangles $I\phi F$ and $I\phi D$, and ϕ is the ortho-centre of $\triangle$ DEF.

$= \phi$ (DEFI).

$\therefore \pi'$, ϕ, D, E, F, I, lie on a conic.

But any conic through the vertices of a $\triangle$ and its ortho-centre is a rectangular hyperbola.

$\therefore \pi'$, ϕ, D, E, F, I, are on a rect. hyperbola.

$\therefore$ The hyperbola (equilateral) through D, E, F, I, passes through π'.

Q. E. D.

Similar relations are also true for the ex-circle.

Question 1179.

(CORRECTED) :—From an external point T (x, y), tangents TP, TQ are drawn to the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ prove that the area of the triangle TPQ is

$$\frac{S^{\frac{3}{2}}\sqrt{-\Delta}}{\Delta - CS},$$

and that the area of the quadrilateral OPTQ, where O is the centre of the conic, is $\frac{\sqrt{-\Delta s}}{C}$

Additional Solution by F. H. V. Gulasekharam.

Let us use Capital letters X, Y for current co-ordinates. Suppose TP = α, TQ = β, $P\hat{T}Q = w$, area of the triangle TPQ = M.

The original equation to the conic is

$$a X^2 + 2h XY + b Y^2 + 2g X + 2f Y + c = 0 \qquad \dots \quad (1)$$

Transferring the origin to T (x, y), the equation takes the form

$$a\,X^2 + 2h\,XY + b\,Y^2 + 2g'\,X + 2f'\,Y + S = 0. \quad \ldots \quad (2)$$

Again transforming the equation, referred to TP, TQ as axes, it takes the form

$$S\left[\left(\frac{X}{\alpha} + \frac{Y}{\beta} - 1\right)^2 + 2\lambda\,XY\right] = 0 \ldots\ldots\ldots\ldots (3)$$

For, the term independent of X and Y in (2) and (3) must be identical.

Now from (1), (2) and (3), the invariant properties give

$$\Delta \sin^2 w = -S^3 \lambda^2 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$C \sin^2 w = -S^2 \lambda \left(\lambda + \frac{2}{\alpha\beta}\right) \quad \ldots \quad \ldots \quad (5)$$

also $\qquad 2M = \alpha\beta \sin w \quad \ldots \quad \ldots \quad (6)$

From (4), (5) and (6) we immediately get

$$M = \frac{S^{\frac{3}{2}} \sqrt{-\Delta}}{\Delta - CS}.$$

Again remembering (i) that the equation to PQ (referred to the original axes) is $x\,(a\,X + h\,Y + g) + y\,(h\,X + b\,Y + f) + (g\,X + f\,Y + c) = 0$, (ii) that T and O are the points (x, y) and $\left(\frac{G}{C}, \frac{F}{C}\right)$ respectively, and (iii) that T and O are on opposite sides of PQ,

$$\text{we get} \quad \frac{\Delta\,OPQ}{\Delta/C} = \frac{\Delta\,TPQ}{-S} = \frac{\text{Quad. } OPTQ}{\frac{\Delta}{C} - S}$$

$$\text{Hence the quad. } OPTQ = \frac{\sqrt{-\Delta S}}{C}.$$

Note:—In the question, as originally proposed by me, the expression for the area of the quadrilateral was wrongly given. The correct expression is given in J. I. M. S. Vol. XIV, p. 37. The original expression

$$\frac{1}{2}\,\frac{\sqrt{-\Delta S}}{\Delta - CS}\left[x\,\frac{\partial S}{\partial x} + y\,\frac{\partial S}{\partial y}\right]$$

gives the area of the quad. O'PTQ, where O' is the origin.

Question 1183.

(B. B. Bagi):—The circles round AQR, BRP, CPQ, where P, Q, R are points in order on the sides BC, CA, AB of a triangle ABC meet in O. If A′, B′, C′ are the middle points of the arcs QOR, ROP, POQ, then show that A′B′C′ is similar to the triangle formed by the ex-centres of ABC. Also show that A′, B′, C′ and the in-centre of ABC are concyclic.

Solution by V. V. S. Narayan and K. Satyanarayana.

We have $\angle C'OB' = \angle C'OP + \angle B'OP = \frac{1}{2}(C + B)$,

$\angle C'IB'$ = Supplement of $\frac{1}{2}(C + B)$, where I is the in-centre.

Hence I, B′, O, C′ are concyclic.

Similarly I, B′, O, A′ can be shown to be concyclic.

Therefore A′, B′, C′, I, O are concyclic.

Consequently the angles A′, B′, C′ of the $\triangle$ A′B′C′ are easily seen to be equal to $\frac{1}{2}(B + C)$, $\frac{1}{2}(C + A)$, $\frac{1}{2}(A + B)$ respectively, which proves that the $\triangle$ A′B′C′ is similar to the ex-central $\triangle$ of ABC.

Note :—The property given in the question is extended by the fact that the point of concurrence O of the circles is also shown to be concyclic with A′, B′, C′, I.

Question 1187.

(F. H. V. Gulasekharam):—If f, g, h be the lengths of the bisectors of the angles A, B, C respectively of a $\triangle$ ABC, prove

$$\text{(i)}\quad [g^2 + h^2 - 2gh\cos\tfrac{1}{2}(B - C)]^{\frac{1}{2}} + [h^2 + f^2 - 2hf\cos\tfrac{1}{2}(C - A)]^{\frac{1}{2}} + [f^2 + g^2 - 2fg\cos\tfrac{1}{2}(A - B)]^{\frac{1}{2}} = 0.$$

$$\text{(ii)}\quad \frac{\sin\frac{1}{2}(B - C)}{f} + \frac{\sin\frac{1}{2}(C - A)}{g} + \frac{\sin\frac{1}{2}(A - B)}{h} = 0.$$

Solution by K. Satyanarayana, A. Mahalingam and the proposer.

Let AF, BG, CH be the bisectors respectively equal to f, g, h.

We shall first prove (ii).

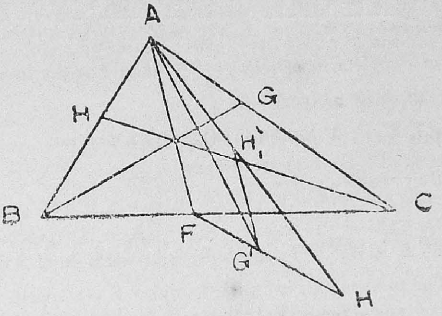
Since
$$f = \frac{2\,bc\cos\frac{A}{2}}{b + c} = \frac{2\triangle}{(b + c)\sin\frac{A}{2}},$$

$$\frac{\sin\frac{1}{2}(B - C)}{f} = \frac{(b + c)\cos\frac{B + C}{2}\sin\frac{B - C}{2}}{2\triangle}$$

$$= \frac{(b + c)(\sin B - \sin C)}{4 \Delta} = \frac{(b + c)(b - c)}{8 \Delta R}$$

$$= \frac{b^2 - c^2}{8 \Delta R}.$$

$$\therefore \sum \frac{\sin \frac{1}{2}(B - C)}{f} = \frac{1}{8 \Delta R} \cdot \sum (b^2 - c^2) = 0.$$



Now, to prove (i), draw AG′, AH′ equal to g, h making angles $\frac{1}{2}(A - B)$, $\frac{1}{2}(A - C)$ with AF. Then by virtue of (ii), it is readily seen that FG′H′ is a straight line.

Further $FG' = \sqrt{(f^2 + g^2 - 2fg \cos\frac{1}{2}(A - B))}$, &c.

Hence (i) is identical with (FG′ + G′H′ + H′F) which is zero, if we take the sign into consideration.

Question 1191.

(G. V. Telang):—ABC is a triangle inscribed in a circle. Three tangents are drawn to the circle so that the portions of them cut off by AB and AC are bisected at the points of contact P, Q and R. Show that the orthocentre of the triangle PQR lies at the mid. point of BC.

Solution by K. Satyanarayana.

Let the centre of the circle be the origin and let A, B, C be the points (0), (α), (β) respectively, the radius of the circle being a.

Let (λ) be the point the portion of the tangent at which intercepted by AB, AC, is bisected at λ.

AB, AC are given by

$$(x + y \tan \frac{\alpha}{2} - a)(x + y \tan \frac{\beta}{2} - a) = 0;$$

or $$x^2 + xy\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}\right) + y^2 \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$-2ax - ay\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}\right) + a^2 = 0. \qquad \ldots (1)$$

Equation of the tangent at 'λ' may be written as

$$\frac{x - a\cos\lambda}{\cos\left(\frac{\pi}{2}+\lambda\right)} = \frac{y - a\sin\lambda}{\sin\left(\frac{\pi}{2}+\lambda\right)} = r,$$

r being distance of (x, y) from λ.

Where this cuts (1), the sum of the values of $r = 0$.

Hence after reduction

$$4\sin^3\frac{\lambda}{2}\cos\frac{\lambda}{2} + \left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}\right)\left(\sin^4\frac{\lambda}{2} - 4\sin^2\frac{\lambda}{2}\cos^2\frac{\lambda}{2}\right)$$

$$+ 2\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\sin\frac{\lambda}{2}\cos\frac{\lambda}{2}\left(1 - 2\sin^2\frac{\lambda}{2}\right) = 0.$$

Taking away the factor $\sin\frac{\lambda}{2}$ and dividing by $\cos^3\frac{\lambda}{2}$, we get

$$(t^3 - 3t)\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}\right) + 2t^2\left(2 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

$$+ 2\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = 0, \text{ where } t = \tan\frac{\lambda}{2}. \qquad \ldots\ldots\ldots(2)$$

(2) is a cubic in t giving 3 values for t, and each value of $\tan\frac{\lambda}{2}$ gives only one value of λ between 0 and 2π. Thus we have 3 points satisfying the condition. Let the three values of λ be λ_1, λ_2 and λ_3.

If t_1, t_2, t_3 be the roots of the Equation

$$\Sigma t_1 = -\frac{2\left(2 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}},$$

$$\Sigma t_1 t_2 = -3;\ t_1 t_2 t_3 = -\frac{2\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}.$$

$$\therefore \quad \tan\left(\frac{\lambda_1+\lambda_2+\lambda_3}{2}\right) = \frac{\Sigma t_1 - t_1 t_2 t_3}{1-\Sigma t_1 t_2}.$$

$$= -\cot\frac{\alpha+\beta}{2} \quad \tan\left(\frac{\pi}{2}+\frac{\alpha+\beta}{2}\right),$$

i. e., $\lambda_1 + \lambda \;\; + \lambda_3 = (2n+1)\pi + \alpha + \beta.$ ... ... (3)

Now the 'm' of the join of λ_1 to mid.-point of BC is

$$\frac{\sin\lambda_1 - \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\cos\lambda_1 - \cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$= \frac{2t_1 - \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}(1+t_1{}^2)}{(1-t_1{}^2) - \cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}(1+t_1{}^2)}$$

while the 'm' of the join of λ_2, λ_3 $= + \tan\left(\frac{\alpha+\beta}{2} - \frac{\lambda_1}{2}\right)$, from (3)

and this reduces to $\dfrac{\sin\frac{\alpha+\beta}{2} - t_1\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + t_1\sin\frac{\alpha+\beta}{2}}.$

Hence the condition for the perpendicularity of the former and the latter is

$$\left[2t_1 - \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}(1+t_1{}^2)\right]\left[\sin\frac{\alpha+\beta}{2} - t_1\cos\frac{\alpha+\beta}{2}\right]$$

$$+\left[(1-t_1{}^2) - \cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}(1+t_1{}^2)\right]$$

$$\left[\cos\frac{\alpha+\beta}{2} + t_1\sin\frac{\alpha+\beta}{2}\right] = 0,$$

or $\quad (t_1{}^3 - 3t_1)\sin\frac{\alpha+\beta}{2} + t_1{}^2\left[3\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha-\beta}{2}\right]$

$$+\left[\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right] = 0$$

which is satisfied since t_1 satisfies (2), and (2) easily reduces to the above form.

Hence the mid.-point of BC is the ortho-centre of the triangles whose vertices are the points λ_1, λ_2, λ_3.

Question 1199.

(F. H. V. GULASEKHARAM) :—Prove that the radius of the pedal circle of a point P with respect to a triangle ABC is given by

$$\frac{R \sin A \sin B \sin C}{\cos PAB \cos PBC \cos PCA + \cos PAC \cos PCB \cos PBA},$$

where R is the circum-radius of ABC. If the pedal triangle of P is in perspective with ABC, prove that the expression above reduces to

$$\tfrac{1}{2}R \; \frac{\sin A \sin B \sin C}{\cos PAB \cos PBC \cos PCA}.$$

Solution by K. Satyanarayana and K. J. Sanjana.

Let us first prove a Lemma: If D, E, F are any three points respectively on BC, CA, AB, then,

$$AF \,.\, BD \,.\, CE + AE \,.\, CD \,.\, BF = 4\,R \,.\, \Delta\, DEF$$

For, denoting BD, CE, AF by x, y, z,

$$\begin{aligned} 4\,R \,.\, \Delta DEF &= 4\,R\,[\Delta\, ABC - \Delta\, AEF - \Delta\, BDF - \Delta\, CDE] \\ &= abc - (b-y)\,za - (c-z)\,xb - (a-x)\,yc \\ &= abc - bcx - cay - abz + ayz + bzx + cxy \\ &= xyz + (a-x)\,(b-y)\,(c-z). \end{aligned}$$

Denoting the pedal Δ in the problem by DEF and its circum-radius by R′,

$$4R' \,.\, \Delta\, DEF = EF \,.\, FD \,.\, DE = PA \,.\, PB \,.\, PC \,.\, \sin A \sin B \sin C,$$

or
$$4\,R \,.\, \Delta\, DEF \,.\, \frac{R'}{R} = (AF \,.\, BD \,.\, CE + AE \,.\, CD \,.\, BF) \,.\, \frac{R'}{R}$$

$$= PA \,.\, PB \,.\, PC \,.\, \sin A \sin B \sin C,$$

or
$$R' = \frac{R \sin A \sin B \sin C}{\frac{AF}{PA} \cdot \frac{BD}{PB} \cdot \frac{CE}{PC} + \frac{AE}{PA} \cdot \frac{BF}{PB} \cdot \frac{CD}{PC}}$$

$$= \frac{R \sin A \sin B \sin C}{\cos PAB . \cos PBC \cos PCA + \cos PAC \cos PCB \cos PBA}.$$

If DEF and ABC be in perspective, AF . BD . CE = AE . BF . CD and hence the second result follows.

Question 1201.

(MARTYN M. THOMAS):—A periodic comet when at an angular distance θ from the perihelion of its orbit, of eccentricity e, suddenly encounters a resistance which brings it to a standstill. Show that it will fall into the sun in time $\frac{t}{4\sqrt{2}}\left(\frac{1-e^2}{1+e\cos\theta}\right)^{\frac{3}{2}}$, where t is the periodic time of the comet.

Solution by M. V. Ramakrishnan.

Since the comet is reduced to rest it moves towards the sun in a straight line. Hence

$$T = \frac{\pi}{2\sqrt{2\mu}} \cdot r^{\frac{3}{2}},$$

where r represents the radius vector to the comet in its position in the orbit when it is reduced to rest.

Now,

$$r = \frac{l}{1+e\cos\theta} = \frac{a(1-e^2)}{1+e\cos\theta},$$

$$\therefore \; T = \frac{\pi}{2\sqrt{2\mu}} a^{\frac{3}{2}} \cdot \left(\frac{1-e^2}{1+e\cos\theta}\right)^{\frac{3}{2}}$$

$$= \frac{2\pi}{\sqrt{\mu}} \cdot a^{\frac{3}{2}} \cdot \frac{1}{4\sqrt{2}} \left(\frac{1-e^2}{1+e\cos\theta}\right)^{\frac{3}{2}}$$

$$= \frac{t}{4\sqrt{2}} \left(\frac{1-e^2}{1+e\cos\theta}\right)^{\frac{3}{2}}.$$

Question 1213.

(V. RAMASWAMI AIYAR) :—Prove that the pedal circle of any point P with respect to a triangle ABC cuts the side BC at an angle equal to the complement of the sum of the angles PAB, PBA, PCB.

Solution by F. H. V. Gulasekharam.

With the notation of J. I. M. S. Vol. XIII, pp. 210—218, from §§ 4 and 5 *ibid*, the perpendicular distance, from BC, of the centre of the pedal circle of P is

$$\frac{\rho'^2}{2a}\left[\cot\beta + \cot\beta' + \cot\gamma + \cot\gamma'\right]$$

$$= \frac{\rho \sin \alpha \sin \beta \sin \gamma}{\sin A} \left[\cot \beta + \cot \beta' + \cot \gamma + \cot \gamma' \right]$$

$$= \rho \left[\frac{\sin \alpha \quad \sin (A + \beta' + \gamma')}{\sin A} + \frac{\sin (A - \alpha) \sin (\beta' + \gamma')}{\sin A} \right]$$

$$= \rho \left[\sin \alpha \ \cos (\beta' + \gamma') + \cos \alpha \ \sin (\beta' + \gamma') \right]$$

$$= \rho \quad \sin (\alpha + \beta' + \gamma').$$

Hence the question.

Note :—The normal co-ordinates of the centre of the pedal circle are proportional to $\sin (\alpha + \beta' + \gamma')$, $\sin (\alpha' + \beta + \gamma')$, $\sin (\alpha' + \beta' + \gamma)$.

Question 1244.

(S. RAJANARAYANAN) :—Sum the series

$$(a-1)_0 \, a_r - a_1 \ a_{r-1} + (a+1)_2 \, a_{r-2} - \ldots (1-1)^r \, (a+r-1)_r a_0$$

where n_r denotes the number of combinations of n things r at a time.

Solution by K. Satyanarayana.

The series is the co-efficient of x^r in

$$a_r \, x_r \, (1+x)^{a-1} - a_{r-1} \, x^{r-1} (1+x)^a + a_{r-2} \, x^{r-2}.$$

$$(1+x)^{a+1} - \ldots + (-1)^r \, a_0 \, x^0 \, (1+x)^{a+r-1}$$

$$i.e., \ (-1)^r \left[(1+x)^{a+r-1} - a_1 \, x \, (1+x)^{a+r-2} + \ldots \right.$$

$$+ (-1)^r . \quad a_r \ x^r \, (1+x)^{a-1} + (-1)^{r+1} \ a_{r+1} \ x^{r+1} \, (1+x)^{a-}$$

$$\left. + \ldots + (-1)^a \ x^a \, (1+x)^{r-1} \right]$$

$$i.e., \ (-1)^r \ (1+x)^{r-1} \ (1+x-x)^a$$

$$i.e., \ (-1)^r \ (1+x)^{r-1}$$

$\therefore$ the sum of the series $= 0$.

Question 1245.

(S. RAJANARAYANAN) :—Find the value of the infinite series

$$S_1 + \frac{S_2}{2!} + \frac{S_3}{3!} + \dots \dots$$

where S_r denotes the sum of the r^{th} powers of the first n natural numbers.

Solution by K. Satyanarayana.

$$\text{The series} = \sum_{r=1}^{r=\infty} \frac{1^r + 2^r + 3^r + \dots n^r}{r!}$$

$$= \sum_{r=1}^{r=\infty} \frac{1^r}{r!} + \sum_{r=1}^{r=\infty} \frac{2^r}{r!} + \dots + \sum_{r=1}^{r=\infty} \frac{n^r}{r!}$$

$$= (e-1) + (e^2-1) + (e^3-1) + \dots + (e^n-1)$$

$$= \left[e.\frac{e^n-1}{e-1} - n\right].$$

Question 1246.

(S. RAJANARAYANAN) :—Find the value of the expression

$$\surd[a + \surd\{ab + \surd(ab^3 + \surd ab^7 + \surd ab^{15} + \surd ab^{31} + \dots)\}]$$

Solution by K. Satyanarayana.

Denoting the value of the expression by x, it is easily seen that

$$x = \surd b \sqrt{\frac{a}{b} + \sqrt{\frac{a}{b} + \sqrt{\frac{a}{b} + \dots\dots}}}$$

$$\text{so that } \frac{x}{\surd b} = \sqrt{\frac{a}{b} + \frac{x}{\surd b}}.$$

$$\text{or} \quad \frac{x^2}{b} = \frac{a}{b} + \frac{x}{\surd b}$$

$$\therefore \quad x = \frac{\surd b}{2} + \sqrt{a + \frac{b}{4}}.$$

QUESTIONS FOR SOLUTION.

1254. (G. S. MAHAJANI) :—At a certain place A, on a certain day,

$$\left.\begin{array}{l} m_0 = \text{mean solar time of sidereal noon,} \\ s_0 = \text{sidereal time of mean noon,} \end{array}\right\}$$

show that on that day the 'mean' and 'sidereal' times (m, s) at any other place B, where L^o is the difference of the longitudes of B and A, are connected by the relation

$$\frac{m \pm L/15}{m_0} + \frac{s \pm L/15}{s_0} = 1.$$

1255. (T. VIJAYARAGHAVAN) :—Show that

$$\lim_{m \to \infty} B\left[m, \frac{\log \log m - \log \log \log m - k}{\log m}\right] = e^k,$$

where $B(p, q)$, as usual, denotes the integral

$$\int_0^1 x^{p-1} (1-x)^{q-1}\, dx.$$

1256. (T. VIJAYARAGHAVAN) :—Let $\theta = a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots\cdots$

be an irrational number and $\lambda(\theta, n)$ be defined by the equation

$$\left| \theta - \frac{p_n}{q_n} \right| = \frac{1}{\lambda(\theta, n) . q_n^2},$$

where $\frac{p_n}{q_n}$ is the nth convergent in the continued fraction ; if

$$k(\theta) = \overline{\lim_{n \to \infty}}\ \lambda(\theta, n),$$

and if 3 occurs an infinity of times in the set (a_r), then show that

$$k(\theta) \geqslant \frac{65 + 9\sqrt{3}}{22},$$

except for a particular class of quadratic surds, namely, those whose periodic part when expressed as a simple continued fraction consists only of 3's.

1257. (A. C. L. WILKINSON).—Solve the partial differential equation

$$r(1-q^2)^2 + 2pqs(1-q^2) + p^2q^2t - (1-q^2)(rt-s^2)z = 0.$$

1258. (A. C. L. WILKINSON):—Solve the partial differential equation

$$xy(1-x^2)r-(2x^2y^2-x^2-y_2+1)s+xy(1-y^2)t+xy(z-px-qy)=0.$$

1259. (P. V. SESHU AIYAR):—Suggested while reading about 'the theory of monopolies' in 'Economics.'

(1) Show that the values of x for which the function $u \equiv xf(x)$ is a maximum are the abscissæ of the points where $y = f(x)$ touches a member of the family of rectangular hyperbolas $xy = c$.

(2) If $y = f(x)$ touches a member of the family $xy = C$ at a point whose abscissa is x_1, show that the curve $y = f(x) - \frac{k}{x}$ where k is a constant touches another member of the family at a point of the same abscissa x_1; and find the 'c' of that other member.

1260. (A. T. THOMAS):—If $a_1 = \sin x$, $a_2 = \sin\sin x$, $a_n = \sin\sin$ (n times) .... x, where $O \angle x \angle \pi$, then prove that the infinite series $\sum_1^\infty a_n$, and $\sum_1^\infty a_n^2$ are divergent, but $\sum_1^\infty a_n^3$ is convergent, and in fact when n is large $a_n \sim \frac{\sqrt{3}}{\sqrt{n}}$.

1261. (A. T. THOMAS):—If the numbers represented by $f(x, y)$ where f is an integral expression with integral co-efficients of the positive integers x,y, are arranged in increasing order of magnitude, and considering those less than N_1 the probability that x and y are prime to each other is a number tending to $\frac{6}{\pi^2}$ as $n \rightarrow \infty$.

1262. (A. T. THOMAS):—If (x) means the fractional part of x, and if x is an irrational number >1, show that such members exist that (x^n) tends to zero steadily as n goes to ∞ through positive integral values.

1263. (I. TOTADRI IYENGAR):—

Sum the series:

$$1-\frac{1-x^m}{1-x}+\frac{(1-x^m)(1-x^{m-1})}{(1-x)(1-x^2)}-\frac{(1-x^m)(1-x^{m-1})(1-x^{m-2})}{(1-x)(1-x^2)(1-x^3)}+\ldots$$

1264. (A. NARASINGA RAO):—A and B are two ovals, C is the envelope of ovals similar to A and similarly placed, having their homothetic centres on B and a constant ratio of similitude λ. Prove that

(i) C is also the envelope of ovals homothetic to B, having the homothetic centres on A and a ratio of similitude $1 - \lambda$;

(ii) if P, Q be points on A, B and the tangents at these points are parallel, C divides PQ in the ratio $(1 - \lambda) : \lambda$;

(iii) if $\rho = f(\psi)$, $\rho = g(\psi)$ be the intrinsic equations of A and B, then that of C is $\rho = \lambda . f(\psi) + (1 - \lambda) . g(\psi)$;

(iv) if masses λ, $(1 - \lambda)$ be placed within or on the boundaries of A and B, their centre of mass will be within or on C; also every interior or boundary point of C is a possible position of centre of mass.

1265. (A. A. Krishnaswami Iyengar):—Show that every odd number can be expressed as the sum of seven squares, except the numbers 1, 3, 5, 9, 11, 17.

1266. (A. A. Krishnaswami Iyengar):—Find general expression for the sides of rational triangles, the squares of whose areas are perfect cubes. (Ex. 5, 6, 7.)

1267. (S. Rajanarayanan):—Prove that, if

$$e^{n} + \frac{1}{n} = e^{m} + \frac{1}{m},$$

then $$e^{m+n} = e^{m} + e^{n},$$ and conversely.

1268. (S. Rajanarayanan):—If $e^{\frac{1}{a}} = e^{\frac{1}{b}} = n^{n^{a}} : n^{n^{b}}$,

then $n^{a+b} = n^{a} + n^{b}$, and conversely.

1269. (R. Vaidyanathaswami):—Solve the equation

$$\frac{dx}{ax^2 + bx + c} = \frac{dy}{a^1 y^2 + b^1 y + c^1}.$$

Hence or otherwise shew that the irreducible algebraic curves which lie on a given quadric and whose tangents belong to a given linear complex, can always be reduced to the form

$$\lambda^{m} = \mu^{n} \ (m \text{ prime to } n)$$

where λ, μ are generator co-ordinates on the quadric [*i. e*, the point (λ, μ) on the quadric is the point the parameters of the generators through which are λ, μ].

Find the invariant relation between the quadric and the complex, in order that one such irreducible curve may be a twisted cubic.